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The Supersymmetric Transfer Matrix for Linear Chains with Nondiagonal Disorder[1]

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A study is made of the supersymmetric transfer matrix of the n-orbital linear chain with Gaussian nondiagonal and diagonal disorder in the matrix (Hubbard–Stratonovich) variables. This formalism is applied to the one-point Green's function. Invariant functions of supersymmetric matrices are discussed in Section 3.

KEY WORDS: Supersymmetric transfer matrix; disordered systems; Green's function.

1. INTRODUCTION

Supersymmetry is expected to provide us with a useful computational tool in several areas of physics. In solid state physics this was confirmed in much work having its roots in the 1979 paper by Parisi and Sourlas[(1)] (for reviews see Refs. 2–4).

The supersymmetric transfer matrix formalism in the spin (vector) variables was first suggested by Efetov[(2)] and was rigorously exploited by A. Klein and co-workers (see, e.g., Ref. 3). Campanino and Klein[(5)] used it to study the density of states for the one-dimensional Anderson model with diagonal disorder, whereas Klein *et al.*[(6)] studied one-dimensional localization.

In this paper we consider the one-dimensional n-orbital model with Gaussian diagonal and nondiagonal disorder proposed by Wegner.[(7)] Defining a supersymmetric transfer matrix after performing the Hubbard–Stratonovich transformation,[(4)] we find a compact formula for the one-point

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Green's function of the model. By integrating out the noncommutative variable, we get an interesting series expansion of the one-point Green's function starting with the Green's function of the zero-dimensional (Wigner) model.

We use this expansion to prove regularity of the density of states, improving some results in Refs. 8, 9, and 13. In one-dimensional models with nondiagonal disorder this is a nontrivial problem. Indeed, it is known that the density of states can be singular at $E=0$,[10–12] because of some strange resonance phenomena. Adding a portion of diagonal disorder to the nondiagonal one restores regularity again.

Finally, we stress that our supersymmetric transfer matrix differs from that in Ref. 5. It includes the Hubbard–Stratonovich transformation of the supersymmetric spin variables used in Ref. 9. In the new dual (matrix) variables this formalism is supposed to apply in the region of extended states of models that exhibit such a regime.[7,13]

2. THE SUPERSPACE ONE-POINT GREEN'S FUNCTION IN THE MATRIX (HUBBARD–STRATONOVICH) VARIABLES

We start with the superspace integral representation of the diagonal matrix elements of the n-orbital one-point function of the unitary ensemble[8,9,13] in the matrix (Hubbard–Stratonovich) variables:

$$G_{00}(\Lambda, E) = -\int \left[\sum_u w_{0u}\alpha(u)\right] \exp\left[-\frac{n}{2} S\,\mathrm{tr}\, QwQ - nS\,\mathrm{tr}\ln(E-Q)\right] DQ \tag{1}$$

where E is the energy ($\mathrm{Im}\, E>0$) and Λ is a finite cube on the lattice $\mathbb{Z}^\nu$ containing the origin. In (1) we have $G=\langle (V-E)^{-1}\rangle$ and $\langle\cdot\rangle$ is the disorder average given by the Gaussian distribution on the Hermitian $n\times n$ matrix V (for details see Refs. 8, 9, and 13):

$$\frac{1}{\mathrm{norm}} \exp\left(-\frac{n}{2}\sum_{\substack{xy\\ \alpha\beta}} \frac{1}{M_{xy}} |V_{xy}^{\alpha\beta}|^2\right) dV \tag{2}$$

In (2), $M_{xy}\geqslant 0$ are matrix elements of a positive-definite, symmetric, and translation-invariant matrix $M=w^{-1}$. A typical example, which will be assumed in this paper, is $M=(-\Delta+m^2)^{-1}$, where Δ is the lattice Laplacian and $m^2>0$. For $\Lambda=\mathbb{Z}^\nu$ and $n\to\infty$, G_{00} reproduces the well-known semicircle law.

The (super) matrix Q is chosen to be

$$Q = \begin{pmatrix} \alpha & \theta \\ \beta & i\gamma \end{pmatrix} \tag{3}$$

In (3), α and γ are (commutative, real) numbers and β and θ are generators of a Grassmann algebra.

We remark that in Ref. 9 we worked with the particular case $\theta = \beta^*$ (complex conjugate of β). This was possible because in Ref. 9 we never used supersymmetry (we used only supervariables). Symmetry arguments require working with a general Q given by (3). Finally, by definition,

$$DQ = \frac{1}{2\pi}\, dQ = \frac{1}{2\pi}\, d\alpha\, d\gamma\, d\theta\, d\beta$$

The imaginary factor in front of γ in Q must be introduced in order to assure convergence in (1).

We remark that the integral representation (1) in the zero-dimensional case (Wigner model) is identical with one given by Brézin.[14,15]

The superspace function under the exponential function in (1) is invariant under a (superspace) conjugacy transformation $Q \to S^{-1}QS$ with S nonsingular. In order to be able to exploit this symmetry, we will study in the next section some supersymmetric *matrix* invariants which recall the Parisi–Sourlas *vector* invariants.[1,3] The content of Section 3 could be of some use for other problems, too.

3. SUPERSYMMETRIC MATRIX INVARIANTS

In this section we will prove some results about invariant functions of 2×2 supermatrices

$$Q = \begin{pmatrix} \alpha & \theta \\ \beta & i\gamma \end{pmatrix}$$

where $\alpha, \gamma \in \mathbb{R}$ and θ and β are two generators of a Grassmann algebra. We find it very convenient to work with the complex variables $z = \alpha + i\gamma$ and $\bar{z} = \alpha - i\gamma$. We say that a function

$$F(Q) = F_0 + F_1\theta + F_2\beta + F_3\theta\beta; \qquad F_i \equiv F_i(z, \bar{z}), \quad i = 0, 1, 2, 3$$

is regular if $F_i(z, \bar{z})$ belongs to $C'(\mathbb{C})$ (continuous differentiable functions) for all $i = 0, 1, 2, 3$ (for the notations and for the z, $\bar{z}$ formalism used in this paper see Ref. 16, Chapter I).

Lemma. Let $F(Q)$ be a regular invariant function of Q. This means that $F(Q)=F(S^{-1}QS)$ for all nonsingular 2×2 supermatrices. Then there exists a regular function $\Phi(z,\bar{z})$ with

$$F(Q)=\Phi(z,\bar{z})+\frac{2}{\bar{z}}\frac{\partial\Phi(z,\bar{z})}{\partial z}\theta\beta \tag{4}$$

where $z=\alpha+i\gamma$, $\bar{z}=\alpha-i\gamma$, and $\Phi(z,\bar{z})=\bar{z}\varphi(z,\bar{z})+C$, where C is a constant. Furthermore, Φ can be recovered from F by taking formally $\Phi=F|_{\theta=\beta=0}$.

Proof. We impose the condition $F(Q)=F(S^{-1}QS)$ for S given by

$$S_1=\begin{pmatrix}1 & 0\\ 0 & b\end{pmatrix},\qquad b\neq 0$$

$$S_2=\begin{pmatrix}1+\frac{1}{2}\hat{\theta}\hat{\beta} & \hat{\theta}\\ \hat{\beta} & 1+\frac{1}{2}\hat{\beta}\hat{\theta}\end{pmatrix},\qquad S_2^{-1}=\begin{pmatrix}1+\frac{1}{2}\hat{\theta}\hat{\beta} & -\hat{\theta}\\ -\hat{\beta} & 1+\frac{1}{2}\hat{\beta}\hat{\theta}\end{pmatrix}$$

where $\hat{\theta}$ and $\hat{\beta}$ are other two generators of the Grassmann algebra to which θ and β belong.

We find using S_1 that $F_1=F_2=0$. Use of S_2 completes the proof, as a lengthy but elementary computation shows.

The following list of examples of invariant functions of matrices will be helpful in Section 4:

(a) $S\operatorname{tr}Q^2=z\bar{z}+2\theta\beta$

(b) $S\operatorname{tr}Q^3=\bar{z}(\frac{1}{4}\bar{z}^2+\frac{3}{4}z^2)+3z\theta\beta$

Here $\Phi(z,\bar{z})=\bar{z}(\frac{1}{4}\bar{z}^2+\frac{3}{4}z^2)$; $(2/\bar{z})\,\partial\Phi/\partial z=3z$.

(c) $S\det(E-Q)^{-1}=\exp[-S\operatorname{tr}\ln(E-Q)]$

$$=1+\frac{2\bar{z}}{2E-z-\bar{z}}+\frac{4}{(2E-z-\bar{z})^2}\theta\beta$$

Here $\varphi=2/(2E-z-\bar{z})$, $C=1$.

(d) $\exp(-AS\operatorname{tr}Q^2)=\exp(-Az\bar{z})-2A[\exp(-Az\bar{z})]\,\theta\beta,\quad A>0$

Here $\Phi=\exp(-Az\bar{z})$, $(2/\bar{z})\,\partial\Phi/\partial z=-2A\exp(-Az\bar{z})$.

(e) $$\Psi(Q_1)=\frac{1}{2\pi}\int\exp(S\operatorname{tr}Q_1Q_2)\,F(Q_2)\,dQ_2 \tag{5}$$

where F is an invariant function of Q_2. Elementary computation gives

$$\Psi(Q_1)=\frac{1}{4\pi i}\int dz_2\, d\bar{z}_2\left(\exp\frac{z_1\bar{z}_2+z_2\bar{z}_1}{2}\right)\left(-\frac{2}{\bar{z}_2}\frac{\partial\Phi}{\partial z_2}+\theta_1\beta_1\Phi\right) \tag{6}$$

We can obtain a nicer formula for Ψ by using the Stokes and Cauchy formulas in the z, $\bar{z}$ variables (see, for instance, Ref. 16, pp. 2–3), i.e., roughly speaking, performing in (6) the integration by parts. One obtains, under the assumption that Φ_F has the right boundary condition at infinity,

$$\Psi(Q_1)=\Phi_F(0)+\frac{1}{4\pi i}\int\left(\exp\frac{z_1\bar{z}_2+z_2\bar{z}_1}{2}\right)\Phi_F(z_2,\bar{z}_2)$$
$$\times\left(\frac{\bar{z}_1}{\bar{z}_2}+\theta_1\beta_1\right)dz_2\, d\bar{z}_2, \qquad \Phi_F(0)\equiv\Phi_F(0,0) \tag{7}$$

Boundary terms disappear because of the boundary condition. This is obviously an invariant function of Q_1 with $C=\Phi_F(0)$.

Before entering the formalism of the (supersymmetric) transfer matrix, we need to know how to multiply two invariants.

Let

$$F(Q)=\Phi_F+\frac{2}{\bar{z}}\frac{\partial\Phi_F}{\partial z}\theta\beta, \qquad B(Q)=\Phi_B+\frac{2}{\bar{z}}\frac{\partial\Phi_B}{\partial z}\theta\beta$$

be two invariant functions. Then we have

$$BF=\Phi_B\Phi_F+\frac{2}{\bar{z}}\frac{\partial}{\partial z}(\Phi_B\Phi_F)\,\theta\beta$$

i.e.,

$$\Phi_{BF}=\Phi_B\Phi_F=B(Q)\,F(Q)\,|_{\beta=\theta=0} \tag{8}$$

where the notation is obvious.

We remark that if $F(Q)$ is a regular invariant function with zero boundary condition at infinity, we immediately obtain from (4)

$$\int F(Q)\,DQ=\Phi_F(0)\equiv F(0)$$

This nice formula, which recalls the Parisi–Sourlas formula,[2,3] is a particular case of a result by F. Wegner.

4. THE SUPERSYMMETRIC TRANSFER MATRIX

Let us consider a linear chain of length $2l$ symmetric around the origin. The one-point Green's function of the linear chain model (with Dirichlet boundary conditions) is given by (1), where Λ is the segment of length $2l$. We take $w = -\Delta + m^2$, $m^2 > 0$, and introduce the transfer matrix (in the Q variables)

$$\hat{T}(Q_1, Q_2) = \exp nS \operatorname{tr} Q_1 Q_2 \tag{9}$$

and the transfer matrix operator

$$(\hat{T}F)(Q_1) = \frac{1}{2\pi} \int \hat{T}(Q_1, Q_2)\, F(Q_2)\, dQ_2 \tag{10}$$

where $F(Q_2)$ is a function of Q_2 such that the integral in (10) makes sense. If F is an invariant function, then $\hat{T}F$ is an invariant function, too, with respect to the conjugancy transformation $Q \to S^{-1}QS$ with S a nonsingular 2×2 supermatrix [for explicit formulas in the case $n = 1$ see example (e) in Section 3]. Taking Neumann or periodic boundary conditions, we can simplify (1) because in these cases

$$\sum_u w_{0u} \alpha(u) = m^2 \alpha(0) \tag{11}$$

A (supersymmetric) computation shows that $m^2\alpha(0)$ can be replaced by $im^2\gamma(0)$.[9]

Now let F be a (regular) invariant function of $Q = Q(-l)$ and $Q = Q(l)$. Then we can write $G_{00}(\Lambda, E)$ with the help of the supersymmetric transfer matrix as

$$G_{00}(\Lambda, E) = -\frac{1}{2} m^2 \int z(0)\, B(0)[(\hat{T}B)^l F]^2\, (0)\, DQ(0) \tag{12}$$

where

$$B(Q) = \exp[-nAS \operatorname{tr} Q^2 - nS \operatorname{tr} \ln(E - Q)] \tag{13}$$

and $A = \frac{1}{2}(2 + m^2)$. In (12) and (13) we have used the notation $B(0) = B(Q(0))$, where $Q(0)$ is the matrix Q at the origin. We can choose $F = 1$, but for the time being we keep working out the general case.

We are going to compute $[(\hat{T}B)^l F](0)$. For simplicity we take first $n = 1$. Introducing the operator $\hat{T}$ as in (10), we have from (7)

$$(\hat{T}F)(Q_1) = \Phi_F(0) + \frac{1}{4\pi i} \int \left(\exp \frac{z_1 \bar{z}_2 + z_2 \bar{z}_1}{2} \right) \Phi_F(z_2, \bar{z}_2) \times \left(\frac{\bar{z}_1}{\bar{z}_2} + \theta_1 \beta_1 \right) dz_2\, d\bar{z}_2 \tag{14}$$

The form of (14) suggests the definition of the new operator on regular functions $\Phi(z, \bar{z})$:

$$(T\Phi)(z_1, \bar{z}_1) = \Phi(0) + \frac{1}{4\pi i} \bar{z}_1 \int dz_2\, d\bar{z}_2 \left(\exp \frac{z_1 \bar{z}_2 + z_2 \bar{z}_1}{2} \right) \frac{\Phi(z_2, \bar{z}_2)}{\bar{z}_2} \tag{15}$$

The rhs of (14) is perfectly defined, because $\bar{z}_2^{-1}$ is integrable with respect to $dz_2\, d\bar{z}_2$. Then

$$(\hat{T}F)(Q_1) = (T\Phi)(z_1, \bar{z}_1) + \frac{2}{\bar{z}_1} \frac{\partial}{\partial z_1} (T\Phi)(z_1, \bar{z}_1)\, \theta_1 \beta_1 \tag{16}$$

and

$$[(\hat{T}B)^l F](0) = (T\Phi_B)^l \Phi_F(0) + \frac{2}{\bar{z}(0)} \frac{\partial}{\partial z(0)} [(T\Phi_B)^l \Phi_F](0)\, \theta(0)\, \beta(0)$$

$$\begin{aligned} [(\hat{T}B)^l F]^2 (0) &= [(T\Phi_B)^l \Phi_F]^2 (0) \qquad\qquad (16') \\ &\quad + \frac{4}{\bar{z}(0)} (T\Phi_B)^l \Phi_F(0) \frac{\partial}{\partial z(0)} [(T\Phi_B)^l \Phi_F](0)\, \theta(0)\, \beta(0) \end{aligned}$$

In order to obtain (16′) from (16), we use the fact that B is defined as the multiplication operator by $B(Q)$ and go inductively. For instance, in the first step, we have

$$\begin{aligned} [(\hat{T}B)^l F](0) &= [(\hat{T}B)^{l-1} (\hat{T}B)F](0) \\ &= [(\hat{T}B)^{l-1}\, \hat{T}BF](0) \\ &= \left\{ (\hat{T}B)^{l-1} \left[T\Phi_B \Phi_F + \frac{2}{\bar{z}(l-1)} \frac{\partial}{\partial z(l-1)} (T\Phi_B \Phi_F) \right.\right. \\ &\quad \left.\left. \times \beta^*(l-1)\, \beta(l-1) \right] \right\} (0) \end{aligned}$$

and so on until we reach (16′).

Finally, the one point function is $(F = 1)$

$$\begin{aligned} G_{00}(\Lambda, E) = &-\frac{1}{4\pi} m^2 \int z(0) \left[\Phi_B(0) + \frac{2\theta(0)\, \beta(0)}{\bar{z}(0)} \frac{\partial}{\partial z(0)} \Phi_B(0) \right] \\ &\times \left([(T\Phi_B)^l \cdot 1]^2 (0) \right. \\ &\left. + \frac{4}{\bar{z}(0)} [(T\Phi_B)^l \cdot 1](0) \frac{\partial}{\partial z(0)} \{[(T\Phi_B)^l \cdot 1](0)\}\, \theta(0)\, \beta(0) \right) \\ &\times dz(0)\, d\bar{z}(0)\, d\theta(0)\, d\beta(0) \\ = &\frac{1}{4\pi} m^2 \int z(0) \left(\frac{4\Phi_B(0)}{\bar{z}(0)} [(T\Phi_B)^l \cdot 1](0) \frac{\partial}{\partial z(0)} \{[(T\Phi_B)^l \cdot 1](0)\} \right. \\ &\left. + \frac{2}{\bar{z}(0)} [(T\Phi_B)^l \cdot 1]^2 (0) \frac{\partial}{\partial z(0)} \Phi_B(0) \right) dz(0)\, d\bar{z}(0) \end{aligned} \tag{17}$$

We apply the Stokes and Cauchy theorems in the z, $\bar{z}$ variables as in Section 3 for $\partial/\partial z(0)$ in front of $\Phi_B(0)$ to obtain

$$G_{00}(\Lambda, E) = -\frac{m^2}{2\pi}\int \frac{\Phi_B(0)}{\bar{z}(0)}\left[(T\Phi_B)^l \cdot 1\right]^2 (0)\, dz(0)\, d\bar{z}(0) \tag{18}$$

This is our compact formula for the one-point Green's function of the linear chain with Gaussian disorder (2) given by the matrix $M = w^{-1}$, $w = -\Lambda + m^2$. It is valid for arbitrary n. The operator T is given by (9), (10), and (15) and $\Phi_B = B\,|_{\theta=\beta=0}$, where $B(Q)$ is given by (13).

The formula (18) is similar to (2.4) in Ref. 5, although in different variables.

5. THE ONE-POINT GREEN'S FUNCTION AND THE DENSITY OF STATES

Our formula (18) for the one-point function has an interesting structure. It can be telescoped as follows.

Let R define the operator on regular functions

$$(R\Phi)(z_1, \bar{z}_1) = \frac{\bar{z}_1}{4\pi i}\int dz_2 \bar{z}_2 \left(\exp\frac{z_1\bar{z}_2 + z_2\bar{z}_1}{2}\right)\frac{\Phi(z_2, \bar{z}_2)}{\bar{z}_2} \tag{19}$$

Then we have from (15)

$$T\Phi = \Phi(0) + R\Phi \tag{20}$$

and $(R\Phi)(0) \equiv (R\Phi)(0, 0) = 0$. It follows that

$$(T\Phi)^l \cdot 1 = \left[I + R\Phi + (R\Phi)^2 + \cdots + (R\Phi)^l\right] \cdot 1 \tag{21}$$

Introducing (21) into (18), we obtain

$$G_{00}(\Lambda, E) = \frac{m^2}{2\pi}\int dz(0)\, d\bar{z}(0)\frac{\Phi_B(0)}{\bar{z}(0)}\left\{\sum_{k=0}^{l}\left[(R\Phi_B)^k \cdot 1\right]\right\}^2 (0) \tag{22}$$

with R given by (19).

For $n = 1$ the first term in the series expansion (22) is

$$G_{00}^{(0)} = -\frac{m^2}{2\pi}\int \frac{\Phi_B}{\bar{z}(0)}\, dz(0)\, d\bar{z}(0) \tag{23}$$

where

$$\Phi_B(Q) = B(Q)\,|_{\theta=\beta=0} = \left(\exp\frac{-(2+m^2)\, z\bar{z}}{2}\right)\left(1 + \frac{2\bar{z}}{2E - z - \bar{z}}\right)$$

The integral in (23) can be performed in the α, γ variables and gives

$$G_{00}^{(0)} = \frac{m^2}{2+m^2} \int \frac{1}{V-E} \, d\rho(V) \tag{24}$$

where

$$d\rho(V) = \frac{1}{\text{norm}} \exp \frac{-(2+m^2)\, V^2}{2} \, dV$$

A similar formula can be obtained for n arbitrary.

In fact, $G_{00}(\Lambda, E)$ from (22) can be expanded as follows:

$$\begin{aligned} G_{00}(\Lambda, E) = G_{00}^{(0)} + \frac{m^2}{2\pi} & \sum_{\substack{k_1, k_2 \geqslant 0 \\ k_1 + k_2 > 0}} \frac{1}{(4\pi i)^{k_1+k_2}} \int \frac{\bar{z}(0)}{\bar{z}(-k_1)\, z(k_2)} \\ & \times \exp\left\{\frac{1}{2} \sum_{k=-k_1}^{k_2-1} [z(k)\, \bar{z}(k+1) + z(k+1)\, \bar{z}(k)]\right\} \\ & \times \prod_{k=-k_1}^{k_2} \Phi_B[z(k), \bar{z}(k)]\, dz(k)\, d\bar{z}(k) \end{aligned} \tag{25}$$

It is well known that the density of states is proportional to the imaginary part of the one-point Green's function. Regularity properties of the density of states for real E for our model with Gaussian diagonal and nondiagonal disorder can be inferred either from (22) by studying the operator R or more directly from (25).

In this paper we will study the convergence of the expansion (25). It turns out that this can be done by estimating just elementary Gaussian integrals. We first write (25) in the α, γ variables:

$$\begin{aligned} & G_{00}(\Lambda, E) \\ & \quad = G_{00}^{0} + \frac{m^2}{2\pi} \sum_{\substack{k_1, k_2 \geqslant 0 \\ k_1 + k_2 > 0}} \frac{1}{(2\pi)^{k_1+k_2}} \int \frac{\alpha(0) - i\gamma(0)}{[\alpha(-k_1) - i\gamma(-k_1)][\alpha(k_2) - i\gamma(k_2)]} \\ & \quad \times \exp\left(-\frac{1}{2} \sum_{k=-k_1}^{k_2-1} \{[\alpha(k) - \alpha(k+1)]^2 + [\gamma(k) - \gamma(k+1)]^2\}\right) \\ & \quad \times \exp\left\{-\frac{m^2}{2} \sum_{k_1 = k_1 + 1}^{k_2 - 1} [\alpha(k)^2 + \gamma(k)^2] - m^2 \alpha(-k_1)^2 - m^2 \alpha(-k_2)^2\right\} \\ & \quad \times \prod_{k=-k_1}^{k_2} \left(1 + \frac{\alpha(k) - i\gamma(k)}{E - \alpha(k)}\right) d\alpha(k)\, d\gamma(k) \end{aligned} \tag{26}$$

In (26) we translate $\alpha(k) \to \alpha(k) - (1/m)i$ and estimate the typical term by absolute value. We estimate the exponential coupling term by one and use $|E - \alpha(k)|^{-1} < m$. After leaving out the integrals in k_1, 0. k_2, the bound of the typical term is a power of the following Gaussian integral:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(m^2/2)(\alpha^2+\gamma^2)} [1 + e^{1/2} m (\alpha^2 + \gamma^2)^{1/2}] \, d\alpha \, d\gamma$$

$$= \int_0^{\infty} r(1 + mr) \, e^{-(m^2/2) r^2} \, dr = \frac{2 + (2\pi e)^{1/2}}{2m^2}$$

The convergence condition $[2 + (2\pi e)^{1/2}]/2m^2 < 1$ gives $m^2 > m_0^2 = 1 + (\pi e/2)^{1/2} \cong 3.06637$.

We realize that a simple estimate by absolute value together with Vitali's theorem provides us with an analytic density of states for values of m^2 of the order of unity [more precisely, for $m^2 > 1 + (\pi e/2)^{1/2}$]. Undoubtedly, this is an improvement over the results in Ref. 9 in which analyticity was obtained (for arbitrary dimensions) only for comparatively very large values of m^2, which, besides typical cluster expansion entropy estimates, were dictated in Ref. 9, Section 3 by making use of Cauchy estimates and the Hadamard inequality for the (fermonic) determinant. In fact, one of the main points of this paper was an explicit computation of the determinant in Ref. 9, Eq. (25), by the technique of supersymmetric transfer matrix. The value of m_0^2 above probably can be slightly improved by using a more appropriate integration contour in the complex α plane, but we do not know if the density of states of the present model is still regular for values of $m^2 > 0$ that are arbitrarily small. Nonrigorous work by the replica method seems to indicate analyticity for all $m^2 > 0$.[10] For $m^2 \leqslant 0$ the matrix $w = -\Delta + m^2$ is no longer positive-definite and the model breaks down.

6. REMARKS AND CONCLUSIONS

We have developed a supersymmetric transfer matrix formalism in the Hubbard–Stratonovich matrix variable. Such matrix variables were introduced with remarkable success by Efetov,[2] Wegner,[13] and Verbaarshot *et al.*[4] in the study of random matrices and random operators on the lattice. In these variables models with disorder make connection to the nonlinear σ-model of statistical mechanics and quantum field theory.

By similar methods to those used in this paper, we can formulate linear singular integral equations describing the linear chain. For the case of a Cayley tree the transfer matrix formalism can also be applied. In the language of integral equations we get nonlinearities. Some singular nonlinear integral equations in superspace describing a yet (from the rigorous point of view) uncontrollable approximation of the two-point function on a Cayley tree were studied by Efetov[17] and Zirnbauer.[18] It would be interesting to extend this study (by using integral equations or even the transfer matrix formalism of this paper) to the genuine Anderson or Wegner model. As a final remark, we mention that the supersymmetric formulas (2.4) and (4.1) in Ref. 5 were obtained by simple recurrence arguments in Ref. 19 from the (nonsupersymmetric) original Anderson model. This method does not work for our model and the supersymmetric derivation route to (18) and (21) seems essential.

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